OSCILLATIONS IN NEARLY-LIAPUNOV SELF-CONTAINED SYSTEMS WITH LAG

PMM Vol. 32, №4, 1968, pp. 567-574

A. F. KLEIMENOV
(Sverdlovsk)

(Received April 10, 1968)

An existence theorem for a periodic solution of a nonlinear nearly-Liapunov self-contained system containing lag in the form of small increments is proved. A practical procedure for the construction of the solution is described.

Equations of the (1.1) type describe mechanical systems which contain materials with an essentially nonlinear elastic characteristic (plastics, rubber) [2], nonlinear dampers with an inelastic restoring force, and systems with a feedback-loop lag (as in locator-type devices [3]), etc.

Pontriagin [4] proved an existence theorem for a periodic solution of a self-contained nearly conservative system without lag.

In the present paper the method of ancillary systems of Shimanov [5 and 6] is used to prove the existence of, and to construct, periodic solutions of Eq. (1.1).

1. Let us consider the system described by differential equations with lag of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{X}(\mathbf{x}) + \mu\mathbf{F}(\mathbf{x}(t), \mathbf{x}(t-\tau), \mu) \tag{1.1}$$

$$\mathbf{x} = \begin{Vmatrix} x_1 \\ \vdots \\ x_n \end{Vmatrix}, \quad \mathbf{X} = \begin{Vmatrix} X_1 \\ \vdots \\ X_n \end{Vmatrix}, \quad \mathbf{F} = \begin{Vmatrix} F_1 \\ \vdots \\ F_n \end{Vmatrix}, \quad A = \begin{Vmatrix} 0 & -\lambda & 0 & \ldots & 0 \\ \lambda & 0 & 0 & \ldots & 0 \\ 0 & 0 & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & a_{n3} & \ldots & a_{nn} \end{Vmatrix}$$

Here λ and a_{ij} are constants; X_k are analytic functions of the variables $x_1, \ldots, x_n$ in the neighborhood of the point $x_1 = \ldots = x_n = 0$ whose expansions in powers of these variables begin with terms of order not lower than two; the functions F_k are analytic in the variables $x_1(t), \ldots, x_n(t), x_1(t-\tau), \ldots, x_n(t-\tau)$ in some neighborhood of the origin, and also in the small parameter μ in the neighborhood of the point $\mu = 0$; τ is the constant lag.

We assume that the generating system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{X}(\mathbf{x}) \tag{1.2}$$

is a Liapunov system, i.e. that the following conditions are fulfilled [1]:

a) the characteristic equation $|a_{ij} - \rho\delta_{ij}| = 0$ does not have zero roots and roots of the form $N\lambda\sqrt{-1}$ (where N is an integer);

b) system (1.2) has a first integral of the form

$$H = x_1^2 + x_2^2 + W(x_3, \ldots, x_n) + S(x_1, \ldots, x_n) = \text{const} \qquad (1.3)$$

Here W is a quadratic form in the variables $x_3, \ldots, x_n$; S is an analytic function of the variables $x_1, \ldots, x_n$ whose expansion begins with terms of order not lower than three.

Dots between vectors, as in $\mathbf{x}\cdot\mathbf{y}$, represent scalar multiplication; the derivative $dH/d\mathbf{x}^\circ$ is computed for generating solution (2.1).

2. We know [1] that under the above assumptions system (1.2) has a periodic solution dependent on the two arbitrary constants c and η,

$$\begin{aligned} x_1^\circ(t+\eta, c) &= c\cos\omega + c^2 x_{12}(\omega) + \ldots \qquad (2.1)\\ x_2^\circ(t+\eta, c) &= c\sin\omega + c^2 x_{22}(\omega) + \ldots\\ x_s^\circ(t+\eta, c) &= c x_{s1}(\omega) + c^2 x_{s2}(\omega) + \ldots \qquad (s = 3, \ldots, n)\\ \omega &= \lambda(t+\eta)[1+h(c)]^{-1}, \qquad h(c) = h_2c^2 + h_3c^3 + \ldots \end{aligned}$$

where $x_{ij}(\omega)$ are periodic functions of ω of period 2π which satisfy the conditions $x_{1j}(0) = x_{2j}(0) = 0$ $(j \geqslant 2)$, and where $h_2, h_3, \ldots$ are some constants of which the first nonzero constant has an even subscript. Series (2.1) converge for all η and for values of c from some neighborhood G of the point $c = 0$. The period of solution (2.1) is given by Formula

$$T = 2\pi/\lambda\,[1 + h(c)] \qquad (2.2)$$

The following theorem is valid.

Theorem. System (1.1) has a periodic solution which becomes a generating solution belonging to family (2.1) for $\mu = 0$ if and only if the parameter c satisfies Eq.

$$R(c) = \int_0^T \mathbf{F}(\mathbf{x}^\circ(t), \ \mathbf{x}^\circ(t-\tau), 0)\cdot\frac{dH}{d\mathbf{x}^\circ}\,dt = 0 \qquad (2.3)$$

Every simple root $c = c_1$ of Eq. (2.3), i.e. every root such that

$$(dR/dc)_{c=c_1} \neq 0 \qquad (2.4)$$

is associated with one and only one periodic solution of system (1.1) analytic in μ in the neighborhood of the point $\mu = 0$. The period of this solution is also an analytic function of μ.

Proof. Let us denote the period of the required periodic solution by $T(1+\mu\alpha)$ where α is generally not equal to zero. Replacing the time t in system (1.1) by $t_1(1+\mu\alpha)$, we obtain

$$\frac{d\mathbf{x}}{dt_1} = [A\mathbf{x} + \mathbf{X}(\mathbf{x}) + \mu\mathbf{F}(\mathbf{x}(t_1), \mathbf{x}(t_1-\tau_1), \mu)](1+\mu\alpha) \qquad (\tau = \tau_1(1+\mu\alpha)) \quad (2.5)$$

The problem reduces to that of finding the periodic solutions of period T of system (2.5). Making the substitution

$$\mathbf{x}(t_1) = \mathbf{x}^\circ(t_1, c) + \mu\mathbf{z}(t_1)$$

in system (2.5), we obtain the following system in $\mathbf{z}$:

$$\frac{d\mathbf{z}}{dt_1} = (A+P)\mathbf{z} + \mathbf{F}^\circ + \alpha\frac{d\mathbf{x}^\circ}{dt_1} + \mu\Phi(t_1, \mathbf{z}(t_1), \mathbf{z}(t_1-\tau_1), \mu) \qquad (2.6)$$

where

$$P = \|p_{ij}\| = \left\|\left(\frac{\partial X_i}{\partial x_j}\right)_0\right\|, \qquad \mathbf{F}^\circ = \mathbf{F}(\mathbf{x}^\circ(t_1), \mathbf{x}^\circ(t_1-\tau_1), 0) \qquad (2.7)$$

$$\Phi = \frac{1}{2} Q(\mathbf{z})\mathbf{z} + \left(\frac{d\mathbf{F}}{d\mathbf{x}}\right)_0 \mathbf{z} + \left(\frac{d\mathbf{F}}{d\mathbf{x}(t_1-\tau_1)}\right)_0 \mathbf{z}(t_1-\tau_1) + \left(\frac{d\mathbf{F}}{d\mu}\right)_0 + \tag{2.7 cont.}$$

$$+ \alpha[(A+P)\mathbf{z} + \mathbf{F}^\circ] + \mu[\ldots], \qquad Q(\mathbf{z}) = \| q_{ij} \| = \left\| \left(\frac{d}{d\mathbf{x}} \left[\frac{\partial X_i}{\partial x_j} \right] \right)_0 \cdot \mathbf{z} \right\|$$

The parentheses $(\ldots)_0$ mean that the generating solution has been substituted in after differentiation. System (2.6) has a periodic solution of period T only if the condition of existence of a periodic soulution of the system obtained from (2.6) for $\mu = 0$ is fulfilled. This condition is [1]

$$\int_0^T \left(\mathbf{F}^\circ + \alpha \frac{d\mathbf{x}^\circ}{dt}\right) \cdot \Psi \, dt = 0 \qquad \left(\Psi = \frac{dH}{d\mathbf{x}^\circ}\right) \tag{2.8}$$

Here the vector function Ψ is the unique periodic solution of period T of the system adjoint to the system in variations for Eqs.(1.2).

Recalling that

$$\int_0^T \alpha \frac{d\mathbf{x}^\circ}{dt_1} \cdot \frac{dH}{d\mathbf{x}^\circ} dt_1 = \alpha \int_0^T \frac{dH}{dt_1} dt_1 \equiv 0$$

we obtain the necessity of condition (2.3).

Let us suppose that condition (2.3) has been fulfilled by the choice of $c = c_1$ and that the point $c = c_1$ lies in the domain G of convergence of series (2.1). We can show that if condition (2.4) is fulfilled, then there exists a unique periodic solution of system (2.6). Along with system (2.6) we consider the ancillary system

$$\frac{d\mathbf{u}}{dt_1} = (A+P)\mathbf{u} + \mathbf{F}^\circ + \alpha \frac{d\mathbf{x}^\circ}{dt} + \mu\Phi + W\varphi \qquad \varphi = \left(\frac{d\mathbf{x}^\circ}{dc} + \frac{t_1}{T}\frac{dT}{dc}\frac{d\mathbf{x}^\circ}{d\eta}\right)_{c=c_1} \tag{2.9}$$

Here the constant W is given by the relation

$$W = -\frac{\mu}{T}\int_0^T \Phi \cdot \Psi \, dt_1 \tag{2.10}$$

The periodic solution of ancillary system (2.9),(2.10) can be found by the method of successive approximations. As the first approximation we take

$$\mathbf{u}^{(1)} = (d\mathbf{x}^\circ / dt_1)_{c=c_1}, \quad W^{(1)} = 0$$

The mth approximation is given by the system

$$\frac{d\mathbf{u}^{(m)}}{dt_1} = (A+P)\mathbf{u}^{(m)} + \mathbf{F}^\circ + \alpha\frac{d\mathbf{x}^\circ}{dt_1} + \mu\Phi^{(m-1)} + W^{(m)}\varphi \tag{2.11}$$

$$W^{(m)} = -\frac{\mu}{T}\int_0^T \Phi^{(m-1)}\Psi \, dt_1, \qquad \Phi^{(m-1)} = \Phi(t_1, \mathbf{u}^{(m-1)}(t_1), \mathbf{u}^{(m-1)}(t_1-\tau_1), \mu)$$

The periodic solution of system (2.11) is of the form

$$\mathbf{u}^{(m)} = M^{(m)}\left(\frac{d\mathbf{x}^\circ}{dt_1}\right)_{c=c_1} + \mathbf{L}\left(t_1, \mathbf{F}^\circ + \alpha\frac{d\mathbf{x}^\circ}{dt_1}\right) + \mathbf{L}(t_1, \mu\Phi^{(m-1)} + W^{(m)}\varphi)$$

Here the operator $\mathbf{L}$ satisfies the same conditions as in [1],(p. 110). The initial system is self-contained, so that we can assume without limiting generality that $u_2(0) = 0$. The constants $M^{(m)}$ can then be determined unambiguously from the condition $u_2^{(m)}(0) = 0$, since $\varphi_2(0) \neq 0$. Under our assumptions concerning the right sides of Eqs.(1.1) we can show (e. g. as in [5]) that for a sufficiently small $|\mu|$ the sequences $\mathbf{u}^{(m)}$ and $W^{(m)}$ converge uniformly to the vector function $\mathbf{u}^*(t_1,\alpha,\mu)$ and to the function $W^*(\alpha,\mu)$.

System (2.6) has a periodic solution if and only if

$$W^*(\alpha,\mu) = -\frac{\mu}{T}\int_0^T \Phi(t_1, \mathbf{u}^*(t_1), \mathbf{u}^*(t_1-\tau_1), \mu)\cdot\Psi\, dt_1 = 0 \qquad (2.12)$$

Recalling the form of the vector function Φ (2.7), we can rewrite Eq.(2.12) in the form

$$W^*(\alpha,\mu) = -(\mu/T)\,[A_1\alpha^2 + B\alpha + C + \mu(..)] = 0 \qquad (2.13)$$

Here

$$A_1 = \int_0^T \left(\frac{1}{2} Q(\mathbf{u}')\mathbf{u}' + (A+P)\mathbf{u}'\right)\cdot\Psi\, dt_1$$

$$B = \int_0^T \left[Q(\mathbf{u}')\mathbf{u}'' + \left(\frac{d\mathbf{F}}{d\mathbf{x}}\right)_0 \mathbf{u}' + \left(\frac{d\mathbf{F}}{d\mathbf{x}(t_1-\tau_1)}\right)_0 \mathbf{u}'(t_1-\tau_1) + (A+P)\mathbf{u}'' + \mathbf{F}^\circ\right]\cdot\Psi\, dt_1$$

$$C = \int_0^T \left[\frac{1}{2} Q(\mathbf{u}'')\mathbf{u}'' + \left(\frac{d\mathbf{F}}{d\mathbf{x}}\right)_0 \mathbf{u}'' + \left(\frac{d\mathbf{F}}{d\mathbf{x}(t_1-\tau_1)}\right)_0 \mathbf{u}''(t_1-\tau_1) + \left(\frac{d\mathbf{F}}{d\mu}\right)_0\right]\cdot\Psi\, dt_1$$

where the vector functions $\mathbf{u}'$ and $\mathbf{u}''$ are periodic solutions of the respective systems

$$\frac{d\mathbf{x}}{dt_1} = (A+P)\mathbf{x} + \frac{d\mathbf{x}^\circ}{dt_1}, \qquad \frac{d\mathbf{x}}{dt_1} = (A+P)\mathbf{x} + \mathbf{F}^\circ \qquad (2.14)$$

We know from general theory [1] that

$$\mathbf{u}' = \frac{1}{\beta}\frac{d\mathbf{x}^\circ}{dc} + t_1\frac{d\mathbf{x}^\circ}{d\eta} \qquad \left(\beta = [1 + h(c)]\frac{dh(c)}{dc}\right) \qquad (2.15)$$

By some simple but quite cumbersome operations we can show that

$$A_1 = 0, \qquad B = \frac{1}{\beta}\frac{dR}{dc}$$

From the theorem on implicit functions we infer that if condition (2.4) is fulfilled, then Eq. (2.13) is uniquely solvable for α, where α is an analytic function $\alpha = \alpha^*(\mu)$ in the neighborhood of the point $\mu = 0$. This determines the period of the periodic solution uniquely. The periodic solution of the initial system (1.1) can be written as

$$\mathbf{x}(t_1) = \mathbf{x}^\circ(t_1 + \eta, c_1) + \mu\mathbf{u}^*(t_1, \alpha^*(\mu), \mu)$$

and is an analytic function of μ in the neighborhood of the point $\mu = 0$. The theorem has been proved.

3. Let us now describe a practical procedure for constructing the periodic solution of system (1.1). Making the time substitution

$$t = t_1(1 + \mu\alpha), \qquad \alpha = \alpha_1 + \mu\alpha_2 + \mu^2\alpha_3 + \ldots$$

in system (1.1), we attempt to find the periodic solution formally in the form of a series

$$\mathbf{x}(t_1, \mu) = \mathbf{x}^\circ(t_1, c) + \mu\mathbf{x}_1(t_1) + \mu^2\mathbf{x}_2(t_1) + \ldots \qquad (3.1)$$

with unknown periodic coefficients $\mathbf{x}_i(t_1)$ of period T.

These periodic coefficients satisfy Eqs.

$$\frac{d\mathbf{x}_i}{dt_1} = (A+P)\mathbf{x}_i + \mathbf{F}^{(i)} + \alpha_i\frac{d\mathbf{x}^\circ}{dt_1} \qquad (3.2)$$

where the vector functions $\mathbf{F}^{(i)}$ are periodic in t_1 with the period T and are entire rational functions in $\mathbf{x}(t_1)$ $\mathbf{x}(t_1 - \tau_1)$.

We can attempt to find the periodic solutions of systems (3.2) by applying the method of ancillary systems directly to systems (3.2) as we did in proving the existence theorem.

However, the following method is simpler and more convenient for practical purposes.

Making the time substitution $t_1 = t_2 [1 + h(c)]$, where $h(c)$ is defined in (2.1), in system (3.2), we obtain

$$\frac{d\mathbf{x}_i}{dt_2} = [(A + P)\mathbf{x}_i + \mathbf{F}^{(i)}][1 + h(c)] + \alpha_i \frac{d\mathbf{x}^\circ}{dt_2} \qquad (3.3)$$

where the vector functions $\mathbf{F}^{(i)}$ are periodic in t_2 with the period $2\pi/\lambda$.

Let us consider system (3.3) for $i = 1$

$$\frac{d\mathbf{x}_1}{dt_2} = [(A + P)\mathbf{x}_1 + \mathbf{F}^\circ][1 + h(c)] + \alpha_i \frac{d\mathbf{x}^\circ}{dt_2} \qquad (3.4)$$

The vector function $\mathbf{F}^\circ$ is defined in (2.7). Along with system (3.4) we consider the ancillary system

$$\frac{d\mathbf{u}_1}{dt_2} = [(A + P)\mathbf{u}_1 + \mathbf{F}^\circ][1 + h(c)] + \alpha_1 \frac{d\mathbf{x}^\circ}{dt_2} + W_{11}\mathbf{\Omega}_1 + W_{21}\mathbf{\Omega}_2 \qquad (3.5)$$

$$W_{k1} = -\frac{\lambda}{2\pi}\int_0^{2\pi/\lambda}\left[(P\mathbf{u}_1 + \mathbf{F}^\circ)[1 + h(c)] + A\mathbf{u}_1 h(c) + \alpha_1\frac{d\mathbf{x}^\circ}{dt_2}\right]\mathbf{\Omega}_k\, dt_2$$

$$\mathbf{u}_1 = \begin{Vmatrix} \mathbf{u}_{11} \\ \cdots \\ \mathbf{u}_{1n} \end{Vmatrix}, \quad \mathbf{\Omega}_1 = \begin{Vmatrix} \cos\lambda t_2 \\ \sin\lambda t_2 \\ 0 \\ \cdots\cdots \\ 0 \end{Vmatrix}, \quad \mathbf{\Omega}_2 = \begin{Vmatrix} \sin\lambda t_2 \\ -\cos\lambda t_2 \\ 0 \\ \cdots\cdots\cdots \\ 0 \end{Vmatrix} \qquad (k = 1, 2)$$

We attempt to find a periodic solution of ancillary system (3.5) in series form,

$$\mathbf{u}_1 = \sum_{\gamma=0}^{\infty} u_1^{(\gamma)} c^\gamma, \qquad W_{k1} = \sum_{\gamma=0}^{\infty} W_{k1}^{(\gamma)} c^\gamma \qquad (k = 1, 2)$$

where $\mathbf{u}_1^{(\gamma)}$ are unknown periodic vector functions of t_2 of period $2\pi/\lambda$; $W_{\lambda 1}^{(\gamma)}$ are unknown constants. As already noted, in computing the periodic solution we can set $u_{12}(0) = 0$ without limiting generality. Hence, as our initial conditions we take

$$u_{11}^{(0)}(0) = \beta_1, \quad u_{12}^{(0)}(0) = 0, \quad u_{11}^{(\gamma)}(0) = u_{12}^{(\gamma)}(0) = 0 \quad (\gamma \geqslant 1) \qquad (3.6)$$

Here β_1 is a constant to be determined.

The periodic solution of system (3.5) with initial conditions (3.6) always exists and can be computed with any degree of accuracy. Since systems (3.5) are linear, and the constants W_{k1} are given by

$$W_{k1} = cE_k(c)\beta_1 + \alpha_1 V_k(c) + H_k^{(1)}(c) \qquad (k = 1,2) \qquad (3.7)$$

Here E_k, V_k, H_k are analytic functions of c, where at least one of the functions $E_1(c)$, $E_2(c)$ does not vanish identically.

The periodic solution of system (3.4) exists if and only if

$$W_{11} = 0, \ W_{21} = 0 \qquad (3.8)$$

The compatibility condition for system (3.8) can be written as

$$R_1(c) = c\,E_1(c)W_{21} - cE_2(c)W_{11} = 0 \qquad (3.9)$$

Condition (3.9) is nothing other than condition (2.3) written in a different form, since the latter is the condition of existence of a periodic solution of system (3.4). Since Eq. (2.3) does not contain α_1, we have a relation

$$S(c) = c\,E_1(c)V_2(c) - cE_2(c)V_1(c) \equiv 0 \qquad (3.10)$$

From now on we assume that condition (3.9) is fulfilled by the choice $c = c_1$. From system (3.8) we find $\beta_1 = \beta_1^*(\alpha_1, c)$. It is clear that $\beta_1^*(\alpha_1, c)$ is an analytic function of c in the neighborhood of the point $c = 0$ with a pole of some order at this point, and that it is a linear function of α_1.

The periodic solution of system (3.4) can be written as

$$\mathbf{x}_1 = \mathbf{u}_1(t_2, \beta_1^*(\alpha_1, c), \alpha_1, c) + M_1\varphi^*(t_2, c) \tag{3.11}$$

where $\varphi^* = (dx^\circ / dt_1)_{c=c_1}$. From the condition $x_{21}(0) = 0$ we find that $M_1 = 0$.

Let us assume now that the vector function $\mathbf{x}_j$ ($j \leqslant m - 1$) have been computed and that they are analytic in the neighborhood of the point $c = 0$ with poles at this point. Let us assume that the quantities $\alpha_1, \ldots, \alpha_{m-2}$ have been determined uniquely and that they are analytic functions at the point $c = 0$ with poles at this point.

Let us consider system (3.3) for $i = m$

$$\frac{d\mathbf{x}_m}{dt_2} = [(A + P)\mathbf{x}_m + \mathbf{F}^{(m)}][1 + h(c)] + \alpha_m \frac{d\mathbf{x}^\circ}{dt_2} \tag{3.12}$$

Here $\mathbf{F}^{(m)}$ are unknown vector functions of t_2, analytic in c in the neighborhood of the point $c = 0$ with poles at this point and analytic in α_{m-1} (the latter is self-evident for $m > 2$; it is easy to show that for $m = 2$ function $\mathbf{F}^{(2)}$ is a linear vector function of α_1). Let us isolate the terms containing α_{m-1} as a factor in $\mathbf{F}^{(m)}$,

$$\mathbf{F}^{(m)} = \alpha_{m-1}\left[\frac{1}{\beta} Q(\mathbf{x}_1)\frac{d\mathbf{x}^\circ}{dc} + \frac{1}{\beta}\left(\frac{d\mathbf{F}}{d\mathbf{x}}\right)_0 \frac{d\mathbf{x}^\circ}{dc} + \frac{1}{\beta}\left(\frac{d\mathbf{F}}{d\mathbf{x}(t_2 - \tau_2)}\right)_0 \frac{d\mathbf{x}^\circ(t_2 - \tau_2)}{dc} + \right.$$
$$\left. + \alpha_1 \frac{1}{\beta}(A + P)\frac{d\mathbf{x}^\circ}{dc} + (A + P)\mathbf{x}_1 + \mathbf{F}^\circ\right] + G^{(m)} \tag{3.13}$$

where Q is defined in (2.7), and where $G^{(m)}$ does not contain α_{m-1}. We rewrite Expession (3.13) as

$$\mathbf{F}^{(m)} = \alpha_{m-1}\left[\frac{1}{\beta}\frac{dP}{dc}\mathbf{x}_1 + \frac{1}{\beta}\frac{d\mathbf{F}}{dc} + \alpha_1\frac{1}{\beta}(A + P)\frac{d\mathbf{x}^\circ}{dc} + (A + P)\mathbf{x}_1 + \mathbf{F}^\circ\right] + G^{(m)} \tag{3.14}$$

Along with system (3.12) we consider the ancillary system

$$\frac{d\mathbf{u}_m}{dt_2} = [(A + P)\mathbf{u}_m + \mathbf{F}^{(m)}][1 + h(c)] + \alpha_m\frac{d\mathbf{x}^\circ}{dt_2} + W_{1m}\mathbf{\Omega}_1 + W_{2m}\mathbf{\Omega}_2 \tag{3.15}$$

$$W_{km} = -\frac{\lambda}{2\pi}\int_0^{2\pi/\lambda}\left[(P\mathbf{u}_m + \mathbf{F}^{(m)})[1 + h(c)] + A\mathbf{u}_m h(c) + \alpha_m\frac{d\mathbf{x}^\circ}{dt_2}\right]\cdot\mathbf{\Omega}_k\,dt_2 \quad (k = 1, 2)$$

Specifying the initial conditions $u_{m1}(0) = \beta_m$, $u_{m2}(0) = 0$, we obtain the periodic solution $\mathbf{u}_m(t_2, \beta_m, \alpha_m, c)$ of system (3.15) and the constants W_{km} in the form of series,

$$\mathbf{u}_m(t_2, c) = \sum_{\gamma=-d}^{\infty}\mathbf{u}_m^{(\gamma)}(t_2)c^\gamma, \qquad W_{km}(c) = \sum_{\gamma=-d}^{\infty}W_{km}^{(\gamma)}c^\gamma$$

where d is the largest order of the pole in the Laurent expansions of the vector functions $\mathbf{F}^{(m)}$. System (3.12) is a periodic solution if and only if

$$W_{km} = cE_k(c)\beta_m + \alpha_1 V_k(c) + H_k^{(m)}(c) = 0 \tag{3.16}$$

Here $H_k^{(m)}(c)$ are analytic functions of c in the neighborhood of the point $c=0$ with poles at this point. The compatibility condition for system (3.16) is similar to condition (3.9),

$$c\,E_1(c)\,W_{2m} - c\,E_2(c)\,W_{1m} = 0 \tag{3.17}$$

Since the $\mathbf{F}^{(m)}$ are linear in α_{m-1}, we can isolate the terms in W_{km} which contain α_{m-1} as a factor,

$$W_{km} = W_{km}'\alpha_{m-1} + W_{km}'' \qquad (k = 1,2)$$

so that condition (3.17) can be written as

$$(cE_1W_{2m}' - cE_2W_{1m}')\,\alpha_{m-1} + (cE_1W_{2m}'' - cE_2W_{1m}'') = 0 \qquad (3.18)$$

Clearly, Eq. (3.18) is solvable for α_{m-1} if

$$cE_1W_{2m}' - cE_2W_{1m}' \neq 0$$

Let us show that

$$\left(\frac{dR_1}{dc}\right)_{c=c_1} = [\beta\,(cE_1W_{2m}' - cE_2W_{1m}')]_{c=c_1} \qquad (3.19)$$

where β is defined in (2.15), and $R_1(c)$ is defined in (3.9). In fact,

$$\frac{dR_1}{dc} = cE_1(c)\,\frac{dW_{21}}{dc} - cE_2(c)\,\frac{dW_{11}}{dc} + \frac{d\,(cE_1(c))}{dc}\,W_{21} - \frac{d\,(cE_2(c))}{dc}\,W_{11}$$

However, for $c = c_1$, $\beta_1 = \beta_1^*(c)$ we have $W_{11} = W_{21} = 0$. Hence,

$$\left(\frac{dR_1}{dc}\right)_{c=c_1} = c_1E_1(c_1)\left(\frac{dW_{21}}{dc}\right)_{\substack{c=c_1\\ \beta_1=\beta_1^*}} - c_1E_2(c_1)\left(\frac{dW_{11}}{dc}\right)_{\substack{c=c_1\\ \beta_1=\beta_1^*}} \qquad (3.20)$$

Moreover, the vector function $\mathbf{u}_1^* = \mathbf{u}_1(t_2, \beta_1, \alpha_1, c)$ is a periodic solution of the ancillary system. Substituting this solution into system (3.5), we obtain identities in c. Differentiating these identities with respect to c, we obtain

$$\frac{d}{dt_2}\left(\frac{d\mathbf{u}_1^*}{dc}\right) \equiv \left[(A+P)\frac{d\mathbf{u}_1^*}{dc}\right][1+h(c)] + \left[\frac{dP}{dc}\mathbf{u}_1^* + \frac{d\mathbf{F}^\circ}{dc}\right][1+h(c)] +$$
$$+ [(A+P)\mathbf{u}_1^* + \mathbf{F}^\circ]\,\frac{dh(c)}{dc} + \alpha_1\frac{d}{dc}\left(\frac{d\mathbf{x}^\circ}{dt_2}\right) + \frac{dW_{11}}{dc}\,\boldsymbol{\Omega}_1 + \frac{dW_{21}}{dc}\,\boldsymbol{\Omega}_2 \qquad (3.21)$$

where the identities are fulfilled for all c.

Comparing identities (3.21) with system (3.15) with allowance for Formula (3.14) and recalling that system (3.15) and relation (3.10) are linear, we obtain

$$(W_{km}')_{c=c_1} = \left(\frac{1}{\beta}\,\frac{dW_{k1}}{dc}\right)_{\substack{c=c_1\\ \beta_1=\beta_1^*}}$$

Hence, on fulfillment of the condition

$$\left(\frac{dR_1}{dc}\right)_{c=c_1} \neq 0 \qquad (3.22)$$

Eq. (3.18) is uniquely solvable for α_{m-1}, where α_{m-1} is an analytic function of c in the neighborhood of $c = 0$ with a pole of some order at this point.

From system (3.16) we obtain $\beta_m = \beta_m^*(\alpha_m, c)$. The periodic solution of system (3.12) is

$$\mathbf{x}_m = \mathbf{u}_m(t_2, \beta_m^*(\alpha_m, c), \alpha_m, c)$$

where $\mathbf{u}_m$ is an analytic vector function of c in the neighborhood of the point $c = 0$ with poles at this point; it is linear in α_m.

Thus, on fulfilling conditions (3.9) and (3.22), we obtain the unique system of formal series (3.1) satisfying system (1.1). The convergence of these series for sufficiently small $|\mu|$ follows from their uniqueness as established in the existence and uniqueness theorem of Section 2.

4. As an example let us consider Eq.

$$\frac{d^2x}{dt^2} + \lambda^2 x - \gamma x^3 - \mu a\,[\lambda^2 x(t-\tau) - \gamma x^3(t-\tau)] = 0 \qquad (4.1)$$

which is a special case of the equation proposed in [2]. The term with the factor μ characterizes the plasticity of a material with a nonlinear elastic characteristic.

The generating equation

$$\frac{d^2x}{dt^2} + \lambda^2 x - \gamma x^3 = 0$$

has the periodic solution [1]

$$x^\circ = c\cos\omega + \frac{c^3\gamma}{32\lambda^2}(\cos\omega - \cos 3\omega) + \frac{c^5\gamma^2}{1024\lambda^4}(23\cos\omega - 24\cos 3\omega + \cos 5\omega) + \ldots$$

$$\omega = \lambda(t+\eta)\left(1 + \frac{3}{8}\frac{\gamma}{\lambda^2}c^2 + \frac{57}{256}\frac{\gamma^2}{\lambda^4}c^4 + \ldots\right)^{-1}$$

with the period

$$T = \frac{2\pi}{\lambda}(1 + h_2c^2 + h_4c^4 + \ldots) = \frac{2\pi}{\lambda}\left(1 + \frac{3}{8}\frac{\gamma}{\lambda^2}c^2 + \frac{57}{256}\frac{\gamma^2}{\lambda^4}c^4 + \ldots\right)$$

We shall attempt to find the solution of Eq. (4.1) in the form

$$x = x^\circ(t_2, c) + \mu\, x_1(t_2, c) + \mu^2 x_2(t_2, c) + \ldots$$

$$t = t_2(1 + h_2c^2 + \ldots)(1 + \alpha_1\mu + \alpha_2\mu^2 + \ldots)$$

where the functions $x_i(t_2, c)$ are periodic in t_2 with the period $2\pi/\lambda$. Condition (3.9) for Eq. (4.1) is

$$R_1(c) = -a\lambda^2 \sin\lambda\tau_2 c + {}^{23}/_{32}\, a\gamma \sin\lambda\tau_2 c^3 + \frac{-27a\sin 3\lambda\tau_2 + 25a\sin\lambda\tau_2}{1024}\frac{\gamma^2}{\lambda^4}c^5 + \ldots = 0$$

$$\tau_2 = \tau(1 + h_2c^2 + \ldots)^{-1}(1 + \alpha_1\mu + \ldots)^{-1}$$

The function $x_1(t_2, c)$ turns out to be

$$x_1 \doteq (2\alpha_1 - a\cos\lambda\tau_2)\left[\frac{2\lambda^2}{3\gamma c}\cos\lambda t_2 - \frac{23}{48}c\cos\lambda t_2 - \frac{1}{16}c\cos 3\lambda t_2\right] + \ldots$$

Computation of the second approximation yields the following equation for determining α_1:

$$-{}^2/_3\, a\lambda^4\gamma^{-1}(2\alpha_1 - a\cos\lambda\tau_2)\sin\lambda\tau_2 c + ({}^{47}/_{24}\, a\alpha_1\lambda^2\sin\lambda\tau_2 - {}^{95}/_{48}\, a\lambda^2 \sin\lambda\tau_2\cos\lambda\tau_2)\,\times$$

$$\times c^3 + \ldots = 0$$

The present study was carried out under the supervision of S. N. Shimanov, to whom the author is sincerely grateful.

BIBLIOGRAPHY

1. Malkin, I. G., Some Problems of the Theory of Nonlinear Oscillations. Gostekhizdat, Moscow, 1956.
2. Sorokin, E. S., Internal friction in materials and structures with nonlinear elastic characteristics. Stroit. Mekh. i Raschet Sooruzh., №3, 1964.
3. Pinney, E., Ordinary Differential-Difference Equations, Isd. inostr. lit., Moscow, 1961.
4. Pontriagin, L. S., Nearly Hamiltonian dynamic systems. Zh. Eksperim. i Teor. Fiz., Vol. 4, №9, 1934.
5. Shimanov, S. N., Oscillations of quasilinear systems with a nonanalytic nonlinearity characteristic. PMM Vol. 21, №2, 1957.
6. Shimanov, S. N., On the theory of quasiharmonic oscillations. PMM Vol. 16, №2, 1952.

Translated by A. Y.